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COMMENT

# An algorithm to generate the polynomial zeros of degree one of the Racah coefficients

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**Abstract.** An algorithm is presented to enumerate the polynomial zeros of degree 1 of the Racah ($6j$) coefficients. This algorithm is based on a Diophantine equation linear in two variables, obtained from a condition satisfied by the parameters in the parametrical formula of Brudno for the polynomial zeros of degree 1 of the $6j$ coefficient.

Recently there has been a considerable interest in the study of the polynomial zeros (also called 'non-trivial' zeros) of the $3j$ and the $6j$ coefficients, particularly those of degree one (Koozekanani and Biedenharn 1974, Beyer *et al* 1985, Srinivasa Rao 1985a, b, Srinivasa Rao and Rajeswari 1984, Vanden Berghe *et al* 1984, Van der Jeugt *et al* 1983).

For the polynomial zeros of degree one, we have obtained (Srinivasa Rao and Rajeswari 1984) closed form expressions, by relating the angular momentum coefficients to formal binomial expansion forms, which were found to be exact for the special case of degree one. Taking the remark of Koozekanani and Biedenharn (1974) seriously, that there could be a connection between the polynomial zeros of the $6j$ coefficient and the algebra of exceptional groups, Vanden Berghe *et al* (1984, Van der Jeugt *et al* 1983) in a series of papers studied the realisations of exceptional Lie algebras of $G_2$, $F_4$ and $E_6$ and accounted for 12 generic zeros. However, it was pointed out by one of us (Srinivasa Rao 1985a, b) that 11 of these generic zeros are polynomial zeros of degree one, which were but a few of the more than one thousand such zeros expressed simply through the aforesaid closed form expression. We have classified (Srinivasa Rao and Rajeswari 1985) the polynomial zeros by their degree.

Brudno (1985) found some formulae for the polynomial zeros of degree one, of the $3j$ and the $6j$ coefficients. He calls these linear zeros. Brudno and Louck (1985) discussed the relation of these zeros to the solutions of Diophantine equations of equal sums of like powers. Brudno's (1985) parametrical formula for the polynomial zeros of degree 1 of the $6j$ coefficient is given by

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ L_1 & L_2 & L_3 \end{Bmatrix} \tag{1}$$

where

$$J_1 = \tfrac{1}{2}(abc + def + adg) - \tfrac{1}{2}$$
$$J_2 = \tfrac{1}{2}(abc + beh + ghi) - \tfrac{1}{2}$$
$$J_3 = \tfrac{1}{2}(adg + beh + def + ghi) - 1$$
$$L_1 = \tfrac{1}{2}(adg + ghi)$$
$$L_2 = \tfrac{1}{2}(beh + def)$$

 

$$L_3 = \tfrac{1}{2}(abc + adg + beh) - \tfrac{1}{2}$$

with the condition:

$$cfi = abc + def + ghi + adg + beh \tag{2}$$

to be satisfied by $a, b, \ldots, i$ which can take any integer values between 1 and $\infty$. The method of arriving at this form (1) consists in equating the $x$ and $y$ in the multiplicative factor $(1 - \delta_{x,y})$ shown by us (Srinivasa Rao and Rajeswari 1984) to the partitioning of a product of nine intergers: *abcdefghi*.

A particular partition leads to (1). These polynomial zeros of degree one have been related by Brudno and Louck (1985) to the solutions of the set of Diophantine equations:

$$\begin{aligned} X^3 + Y^3 + Z^3 &= U^3 + V^3 + W^3 \\ X + Y + Z &= U + V + W. \end{aligned} \tag{3}$$

Having established this, they proceed to prove that the enumeration of the polynomial zeros of degree 1, for the $6j$ coefficient, given by Brudno (1985) is complete.

We have already shown (Srinivasa Rao and Rajeswari 1985) that, with the simple closed form expression $(1 - \delta_{x,y})$, we can enumerate all the polynomial zeros of degree 1. We now show that the same list can be generated using the parametric formula (1) of Brudno (1985), with the help of a very simple algorithm. The algorithm makes use of the single linear Diophantine equation:

$$\alpha pq = \beta p + \gamma q + \delta \tag{4}$$

which was solved by Brahmagupta in the sixth century AD (see Dickson 1952). To solve (4), let $\varepsilon$ be an integer and let $\zeta = (\alpha\delta + \beta\gamma)/\varepsilon$. Choosing only those integer values of $\varepsilon$ which will give integer values of $\zeta$, the solutions for $p$ and $q$ in (4) are given by

$$\frac{1}{\alpha}[\max(\varepsilon, \zeta) + \min(\beta, \gamma)] \qquad \frac{1}{\alpha}[\min(\varepsilon, \zeta) + \max(\beta, \gamma)] \tag{5a}$$

$$\frac{1}{\alpha}[\max(\varepsilon, \zeta) + \max(\beta, \gamma)] \qquad \frac{1}{\alpha}[\min(\varepsilon, \zeta) + \min(\beta, \gamma)]. \tag{5b}$$

In these sets ($5a$) and ($5b$), $p$ is that containing $\gamma$. Recalling that the integers $a, b, \ldots, i$ satisfying (2) can take values from 1 to $\infty$, our algorithm is the following.

(i) Choose $a$, $b$, $d$, $e$, $g$ and $h$ to have values 1-10 (say), successively and arrange these into a nest of loops.

(ii) Choose a value of $i$, also to be 1-10 (say).

(iii) The conditional equation (2) reduces to the linear Diophantine equation in the two unknowns $c$ and $f$ (corresponding to $p$ and $q$) in equation (4). The integer solutions of this equation are given by (5).

This algorithm has been successfully tested by us (Srinivasa Rao and Rajeswari 1986) on a computer to generate *all* the required polynomial zeros of degree 1, up to any desired values of the arguments of the $6j$ coefficient.

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